Moscow Institute of Physics and Technology
(National Research University)

Higher School of Modern Mathematics

As a manuscript

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Modelling of logical systems by means of their fragments

RESUME OF THESIS

for the purpose of obtaining academic degree
Doctor of Phisical and Mathematical Science

Moscow – 2025

Introduction

The research carried out by the author is concerned with the expressiveness of languages, logics, and theories, and above all with the algorithmic expressiveness (including computational complexity) of certain fragments of thereof.

Many natural logical systems are either algorithmically undecidable (and sometimes highly undecidable), or, being decidable, have a high complexity of the decision problem. It is known that certain restrictions imposed on the means of the language or the semantics used lead to a change in the algorithmic complexity of some problems. At the same time, sometimes this is not the case: for example, in non-classical logics, both the undecidability and the high complexity of the decision problem in the case of decidability can be obtained with very tight restrictions on the means of the language.

It seems relevant not only to find the boundaries within which such problems turn out to be algorithmically easy or, conversely, remain algorithmically hard, but also to develop general methods that allow us to obtain estimates of the algorithmic complexity of fragments not only of individual logical systems, but of all systems of certain infinite classes. Together with the methods, we would like to have common features or criteria that allow us to immediately obtain the algorithmic complexity of certain fragments of the system we are interested in, or at least about the possibility or impossibility of using these methods.

Similar studies have been carried out both in the field of propositional systems and in the field of predicate logics and theories. In this thesis, approaches and methods are proposed that allow, in particular, to answer many questions about the algorithmic complexity of fragments of various systems that remained open in the publications of researchers who studied them.

1. Research area

The decision problem for many "standard" modal propositional logics is PSPACE-complete [11, 34, 88]; the same applies to the intuitionistic propositional logic and some closely related logics [10, 75]. In addition, for some modal logics, it was proven that even their one-variable fragments are PSPACE-complete [7, 22, 74, 77].

The computational complexity of the decision problem for such logics turned out to be closely related to the issues of the complexity of their approximation by Kripke frames. Questions about the complexity of approximation of non-classical logics were posed by A. V. Kuznetsov [32] in the context of the intuitionistic logic. At the same time, A. V. Kuznetsov himself assumed that the intuitionistic logic is approximable by polynomial-size Kripke frames, and therefore, decidable in polynomial time, and even proposed a construction that allows one to embed the intuitionistic logic into the classical logic by a polynomial-time algorithm, assuming this hypothesis is true [33]. Later M. V. Zakharyaschev and S. V. Popov showed that the intuitionistic logic is not approximable by polynomial-size Kripke frames [87]; the same is true for modal logics, too.

Nevertheless, the one-variable fragment of the intuitionistic logic is decidable in polynomial time, which follows from the construction of Rieger–Nishimura [41, 43]. The complexity of fragments with a larger number of variables was unknown [11, Problem 18.4], and A. V. Chagrov put forward a hypothesis that each of these fragments is approximable by polynomial-size Kripke frames, and hence, decidable in polynomial time.

The author of the thesis began researching these issues under the supervision of A. V. Chagrov in 2001. In the course of research, results were obtained concerning the complexity of the decision problem for infinite families of non-classical logics; in addition, both for logics in the full language and for their fragments in a language with a finite number of variables, results were obtained concerning the complexity of their approximation by Kripke frames, as well as the relationship of the complexity of approximation to the computational complexity of the decision problem. The methods that developed during the research were then transferred to polymodal propositional logics [3, 46, 47, 54, 57, 64, 66, 67] and predicate logics [2, 28, 29, 45, 49, 52, 55, 59, 60, 62, 63, 65].

The classical predicate logic **QCl** is undecidable [13,81] (more precisely, Σ^0_1-complete [23]). The area of research on the decidability of its fragments is known as the "classical decision problem" [9]: there are a lot of results that give both decidable and undecidable fragments of **QCl**, classical first-order theories, as well as other formal predicate systems [17,19,20,25–27,30,31,38,39,76,86]. So, to prove the undecidability of **QCl**, it is sufficient for its language to contain one binary predicate letter and three individual variables [78]. At the same time, the following fragments of **QCl** are decidable:

- the monadic fragment, even with equality [8];
- various guarded fragments [19, 39];
- the fragment with two individual variables [20, 38].

Many first-order theories are also undecidable, in particular, theories in the language with only one binary predicate letter: for example, the theory of a symmetric irreflexive binary relation (that can be extracted from the constructions proposed in [30, 40]) and, as a consequence, the theory of a symmetric reflexive binary relation. As for theories of classes of finite models, they may not even be recursively enumerable [79, 80] (similar results for other languages can be found, for example, in [6, 21]).

The undecidability of modal predicate logics instantly follows from the undecidability of the classical predicate logic. To prove the undecidability of the classical predicate logic, predicate letters of arity of at least two, as well as at least three individual variables, are required. Therefore, the question arises about the algorithmic complexity of monadic fragments of modal predicate logics, as well as their fragments with at most two individual variables.

Fragments with a single individual variable can be decidable: as was noted by D. M. Gabbay and V. B. Shehtman [17], the decidability of the logic fragment **QS5** with one individual variable follows from [73], and a similar result for **QS4** follows from [15]; in addition, in [4] the decidability of a fragment of **QGL** with a single individual variable is proven. Therefore, the question of the decidability of fragments with two individual variables, in particular, of monadic fragments, is of especial interest.

It is known that, unlike the monadic fragment of the classical predicate logic, the monadic fragments of modal predicate logics are undecidable in many "natural" cases, and for the corresponding proof it is sufficient for their languages to contain two unary predicate letters and infinitely many individual variables [31] or two individual variables and infinitely many unary predicate letters [17, 27].

It follows from the undecidability of the classical predicate logic **QCl** that the intuitionistic predicate logic **QInt** is undecidable too, since **QCl** is embeddable into **QInt**. It is known that the monadic fragment of **QInt** is undecidable, even with one unary predicate letter in the language [16, 35], and it is also known that the fragment of **QInt** with two individual variables is undecidable [17,27]. The decidability of the fragment of **QInt** with one individual variable follows from the decidability of the corresponding fragment of **QS4**.

The present thesis shows that these results on undecidability can be strengthened and extended to a large class of logics, including both modal and superintuitionistic predicate logics, and extensions of predicate variants of basic and formal Visser logic. At the same time, general methods are proposed that allow modelling binary predicate letters with unary ones (in the spirit of the Kripke method [31] for modal logics), as well as unary predicate letters by formulas with a single unary predicate letter.

In addition, in the thesis, a lot of attention is paid to the algorithmic complexity of logics of non-elementary classes of Kripke frames. It is known that if a logic (any, including modal, superintuitionistic, propositional, predicate) is complete with respect to an elementary class of Kripke frames, then it is embeddable into the classical predicate logic [18,53,84], and therefore it is recursively enumerable. If the logic is Kripke complete, but not complete with respect to an elementary class of Kripke frames, then such logic is often not recursively enumerable.

There are non-elementary classes of Kripke frames, including infinite frames, whose modal predicate logics are of particular interest. For example, such "natural" classes include classes of finite Kripke frames of a particular logic, various classes of trees, as well as classes of frames of such logics as **QGL**, **QGrz**, **QwGrz**, and others.

The thesis examines logics defined by such classes of Kripke frames, and demonstrates methods that make it possible to prove that these logics are highly undecidable even in very poor languages. This allowed us to extract some corollaries, including those concerning the Kripke completeness of various calculi and their fragments.

2. Main results submitted for defense

The thesis presents results on the algorithmic complexity of fragments of both propositional [47, 48, 57, 58, 66, 70] and predicate [49, 51, 52, 59–63, 68, 69] logics. The standard notation for logics and formulas is used below [11, 18]; the sign "+" is understood as the union of sets of formulas followed by closure under modus ponens, generalization and substitution, and the sign "$\oplus$" as the union of sets of formulas followed by closure under modus ponens, generalization, substitution and necessitation.

Theorem 1 (the result is presented in [48]). *Let* $\mathbf{Int} \subseteq L \subseteq \mathbf{KC}$. *Then the positive fragment of* L *with two variables is* PSPACE*-hard.*

Theorem 2 (the result is presented in [66]). *Products and semiproducts of logics with the first factor* **K**, **T**, **KB** *or* **KTB**, *are embeddable into their one-variable fragments in polynomial time.*

This makes it possible to obtain a number of results on the algorithmic complexity of fragments with a finite number of variables for large classes of logics, taking into account the results on the complexity of products of logics in the entire language [24, 36].

Theorem 3 (the result is presented in [66]). *Let* $\mathbf{K} \times \mathbf{K} \times \mathbf{K} \subseteq L \subseteq \mathbf{KTB} \times \mathbf{S5} \times \mathbf{S5}$. *Then the one-variable fragment of* L *is undecidable.*

Theorem 4 (the result is presented in [66]). *Let* $\mathbf{K} \times \mathbf{K} \subseteq L \subseteq \mathbf{KTB} \times \mathbf{S5}$. *Then the one-variable fragment of* L *is* coNEXPTIME*-hard.*

The last statement has the following refinement concerning some special logics.

Theorem 5 (the result is presented in [66]). *Then the one-variable fragments of* $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{K4}$, *and* $\mathbf{K} \times \mathbf{S5}_2$ *are non-elementary.*

The following results are obtained for dynamic propositional logics.

Theorem 6 (the result is presented in [57]). *The variable-free fragment of* **PDL** *is* EXPTIME*-complete, even in the language with a single atomic program and iteration only.*

Theorem 7 (the result is presented in [57]). *The variable-free fragment of* **IPDL** *is* 2EXPTIME*-complete, even in the language with a single atomic program.*

Theorem 8 (the result is presented in [57]). *The variable-free fragment of* **PRSPDL** *is undecidable.*

The used technique of simulating variables by formulas with a fixed finite number of variables allows us to obtain an exponential lower bound of the complexity function for the corresponding fragments of the PSPACE-hard monomodal logics.

Nevertheless, there is no direct connection between the complexity function of a logic and the computational complexity of the decision problem for it.

Theorem 9 (the result is presented in [70]). *For every degree of unsolvability, there is a normal linearly approximable extension of* **K4** *belonging to the degree whose variable-free fragment also belongs to the degree.*

Theorem 10 (the result is presented in [70]). *Let* $L \in \{\mathbf{KTB}, \mathbf{GL}, \mathbf{Grz}\}$. *For every degree of unsolvability, there is a normal linearly approximable extension of* L *belonging to the degree whose one-variable fragment also belongs to the degree.*

Theorem 11 (the result is presented in [70]). *For every degree of unsolvability, there is a normal linearly approximable extension of* **Int** *belonging to the degree whose two-variable fragment also belongs to the degree.*

Now, let us describe the main results obtained for modal and superintuitionistic predicate logics.

Theorem 12 (the result is presented in [49]). *Let* L *be a modal predicate logic containing* **QCl** *and contained in* **QS5**, $\mathbf{QGL} \oplus \boldsymbol{bd}_2 \oplus \boldsymbol{bf}$, $\mathbf{QGrz} \oplus \boldsymbol{bd}_2 \oplus \boldsymbol{bf}$, $\mathbf{QGL.3} \oplus \boldsymbol{bf}$ *or* $\mathbf{QGrz.3} \oplus \boldsymbol{bf}$. *Then the fragment of* L *in the language with a single unary predicate letter is algorithmically undecidable.*

This result was later improved for a lot of logics. Exactly, the following theorem was obtained.

Theorem 13 (the result is presented in [59]). *Let* L *be a modal predicate logic containing* **QCl** *and contained in* **QKTB**, $\mathbf{QGL} \oplus \boldsymbol{bf}$ *or* $\mathbf{QGrz} \oplus \boldsymbol{bf}$. *Then the fragment of* L *in the language with a single unary predicate letter and two individual variables is algorithmically undecidable.*

It was essential in the proof that all the logics have arbitrarily large (and even infinite) frames. The following observations show that this condition is indeed important.

Theorem 14 (the result is presented in [49]). *Let* L *be a modal predicate logic defined by a finite Kripke frame. Then the monadic fragment of* L *is algorithmically decidable.*

Theorem 15 (the result is presented in [71]). *For every* $n \in \mathbb{N}$, *the monadic fragments of* $\mathbf{QAlt}_n$, $\mathbf{QTAlt}_n$, $\mathbf{QAlt}_n \oplus \boldsymbol{bf}$, *and* $\mathbf{QTAlt}_n \oplus \boldsymbol{bf}$ *are algorithmically undecidable.*

Similar results are obtained for the monadic fragments of modal predicate logics with equality.

In general, in the case of logics of infinite classes of finite Kripke frames, the situation with decidability is different.

Theorem 16 (the result is presented in [60]). *Let L be a modal predicate logic containing* **QCl** *and contained in* **QS5**, **QGL.3** *or* **QGrz.3**. *Then the fragments of L_{wfin} and $L_{wfin} \oplus \boldsymbol{bf}$ in the language with three individual variables are Σ^0_1-hard and Π^0_1-hard.*

Note that in this case, we can additionally require that the language of L does not contain predicate letters, except a single binary letter.

The following result was obtained for the monadic fragments of modal predicate logics of classes of finite frames.

Theorem 17 (the result is presented in [60]). *Let L be a modal predicate logic containing* **QCl** *and contained in* **QKTB**, **QGL** *or* **QGrz**. *Then the fragments of L_{wfin} and $L_{wfin} \oplus \boldsymbol{bf}$ in the language with a single unary predicate letter and three individual variables are Π^0_1-hard.*

Similar results were obtained for superintuitionistic predicate logics, as well as the predicate counterparts of the basic and formal Visser logics [85], which we denote **QBL** and **QFL**, respectively.

Theorem 18 (the result is presented in [59]). *Let* **QBL** $\subseteq L \subseteq$ **QKC** *or* **QBL** $\subseteq L \subseteq$ **QFL**. *Then the positive fragment of L is algorithmically undecidable in the language with a single unary predicate letter and two individual variables.*

For superintuitionistic logics of classes of finite Kripke frames, results similar to the modal case are obtained.

Theorem 19 (the result is presented in [63]). *Let L be a superintuitionistic predicate logic contained in* **QLC**. *Then the positive fragments of L_{wfin} and $L_{wfin} + \boldsymbol{cd}$ in the language with three individual variables are Σ^0_1-hard and Π^0_1-hard.*

Theorem 20 (the result is presented in [63]). *Let L be a superintuitionistic predicate logic, contained in* **QKC**. *Then the positive fragments of L_{wfin} and $L_{wfin} + \boldsymbol{cd}$ in the language with a single unary predicate letter and three individual variables are Π^0_1-hard.*

The methods used can also be extended to logics defined by other classes of frames that are not elementary definable. The thesis presents the results of a study of the algorithmic properties of modal predicate logics of some linear orders, as well as Noetherian orders.

Theorem 21 (the result is presented in [62]). *Let $\alpha = \omega \cdot m + k$ be an ordinal, where $1 \leqslant m < \omega$ and $k < \omega$. Let R be a binary relation on α lying between $<$ and $\leqslant$, where $<$ is the natural strict order on α and $\leqslant$ is its reflexive closure. The modal predicate logic of the Kripke frame $\langle \alpha, R \rangle$ is Π^1_1-hard in the language with a single unary predicate letter, single proposition letter, and two individual variables.*

Note that the simulation used allows us to simultaneously obtain results concerning the algorithmic complexity of the monadic fragments of logics of Kripke frames $\langle \mathbb{Q}, < \rangle$, $\langle \mathbb{Q}, \leqslant \rangle$, $\langle \mathbb{R}, < \rangle$, $\langle \mathbb{R}, \leqslant \rangle$, and some others. Namely, the logics of such frames are Σ^0_1-hard in the language with a single unary predicate letter, a single proposition letter, and two individual variables. It is known [14] that the logic of the Kripke frame $\langle \mathbb{Q}, \leqslant \rangle$ is **QS4.3** and the logic of the Kripke frame $\langle \mathbb{Q}, < \rangle$ is **QK4.3.D.X**, therefore, we conclude that the specified fragments of these logics are Σ^0_1-complete.

Now let us turn to the results obtained for logics of Noetherian orders. For the modal predicate logic L, we define the modal predicate logic L^* as the logic of the class of Kripke frames of L.

Theorem 22 (the result is presented in [52]). *Let L be a modal predicate logic such that $\mathbf{QwGrz}^* \subseteq L$ and either $L \subseteq \mathbf{QGL.3}^* \oplus \boldsymbol{bf}$ or $L \subseteq \mathbf{QGrz.3}^* \oplus \boldsymbol{bf}$. Then the fragment of L in the language with a single unary predicate letter, a single proposition letter, and tree individual variables is Π^1_1-hard.*

Theorem 23 (the result is presented in [52]). *Let L be a modal predicate logic such that $\mathbf{QwGrz}^* \subseteq L$ and either $L \subseteq \mathbf{QGL.3}^* \oplus \boldsymbol{bf}$ or $L \subseteq \mathbf{QGrz.3}^* \oplus \boldsymbol{bf}$. Then the fragment of L in the language with two unary predicate letters, a single proposition letter, and two individual variables is Π^1_1-hard.*

Theorem 24 (the result is presented in [52]). *Let L be a modal predicate logic such that $\mathbf{QwGrz}^* \subseteq L$ and either $L \subseteq \mathbf{QGL}^* \oplus \boldsymbol{bf}$ or $L \subseteq \mathbf{QGrz}^* \oplus \boldsymbol{bf}$. Then the fragment of L in the language with a single unary predicate letters, a single proposition letter, and two individual variables is Π^1_1-hard.*

As a result, we obtain that every recursively enumerable logic between **QwGrz** and $\mathbf{QGL.3} \oplus \boldsymbol{bf}$, as well as between **QwGrz** and $\mathbf{QGrz.3} \oplus \boldsymbol{bf}$ is Kripke incomplete (Kripke incompleteness of some of these logics was obtained earlier in [37] by other methods), and even their specified fragments.

Taking into account these results, the question arises about the existence of recursively enumerable Kripke complete modal predicate logics that are not definable by elementary classes of Kripke frames. Some examples of such logics have been constructed.

Theorem 25 (the result is presented in [61]). *There are Kripke complete recursively enumerable modal predicate logics, which are not complete with respect to elementary definable classes of frames.*

3. Description of methods used

Both common and special methods were used. Common methods include various general mathematical methods (for example, mathematical induction), as well as syntactic and semantic methods (for example, Kripke semantics was used a lot), methods of theory of algorithms, methods of computational complexity theory, methods of lattice theory, methods of theory of tilings. We do not give detailed explanations of these methods, and focus in more detail on special methods.

- The method proposed by J. Halpern was used [22], which allows us to model propositional variables of a language with formulas containing a single propositional variable. This method was first generalized in the class of propositional logics [47, 57, 58, 66] (in particular, in some cases, such modelling was obtained using variable-free formulas), and then modified and applied in the class of modal predicate logics [2, 52, 59, 60, 65].
- The method, proposed by P. Blackburn and E. Spahn [7], of modelling all variables of a language by formulas with a single variable was used. Modifications of this method have proven themselves well in studies related to modal predicate logics defined by Kripke linear frames [62].
- A method has been proposed for modelling propositional variables of an intuitionistic language with formulas containing two variables only [48]. This method was then transferred to the superintuitionist predicate logics [60, 63].
- The method of S. Kripke [31] was used and modified, which allows us to simulate binary predicate letters in classical predicate formulas using modality and unary predicate letters [49]. This method has been transferred to various classes of modal and superintuitionistic predicate logics, as well as to predicate variants of basic and formal Visser logic [60, 63].
- Methods of modelling tiling problems [5, 23] were used, described in the works of F. Walter, M. Zakharyaschev, R. Kontchakov, and A. Kurucz [27, 86], with the help of modifications of which results on the undecidability and strong undecidability of modal and superintuitionist predicate logics were obtained [52, 62].
- The standard translation of modal predicate formulas into formulas of the classical predicate language [18, 53, 84] was used, which allowed us to obtain one of the results of the paper [61].

- The method of proving the decidability of the monadic fragment of classical predicate logic has been modified [8], which allowed us to obtain an upper bound on the algorithmic complexity of non-classical predicate logics defined by classes of finite Kripke frames [49].

4. Areas of possible application of the results

The results and methods of the research can be transferred to other classes of logics. Here are some examples.

Let us begin with propositional logics. Although the results concerning such logics are not submitted for defense, they are contained in the thesis (works [47, 48, 57, 58, 66, 70]), and it is quite reasonable to talk about the further application of appropriate methods.

- It would be interesting to extend the described methods to the relevant [44, 82, 83] and intuitionistic modal logics [15, 42].
- It would be interesting to identify such situations and such logics when methods of modelling all variables of a language with formulas containing a fixed finite number of variables do not work, and also to understand what the complexity of the decision problem is in each case.

At the same time, some studies have already been carried out.

- It was possible to extend the results about PSPACE-hardness of variable-free fragments of modal logics [12] to the class of all logics lying between **K** and **wGrz** [1].
- Arbitrarily hard linearly approximable logics are found in the classes of non-normal and quasi-normal logics [50], as well as in classes of extensions of basic and formal Visser logics.
- Polynomial embeggings of modal intuitionistic logics into their fragments with one variable are constructed [56].

Now let us turn to predicate logics. There are quite a lot of questions left outside the scope of the scientific papers included in the thesis.

- In the class of non-classical predicate logics, there are unresolved questions about the algorithmic complexity of "standard" logics in a language with a single individual variable, the logics of classes of finite Kripke frames in a language with two individual variables, as well as many other logics defined by "natural" elementary indefinable classes of Kripke frames. The results of the work show that different things can be expected here: some of such logics can be recursively enumerable in the full language, while some other can be strongly undecidable in a rather poor fragment of the language.

- It has been proven (but there are no publications at the time of writing this text) that the set of valid first-order formulas in a language with a binary predicate letter and three individual variables is recursively inseparable from the set of formulas in the same language, which are refuted on the class of finite models, where the binary predicate letter is interpreted by a symmetric irreflexive binary relation. This makes it possible to significantly expand the class of modal and superintuitionistic predicate logics, for which it is known that their fragment with a single unary predicate letter and three individual variables is undecidable.
- V. B. Shehtman raised the question on the recursive separability of the monadic fragments of modal predicate logics and the complements of the logics defined by the classes of finite Kripke frames of the original logics. The methods developed in the thesis allow us to explore this issue.
- The ideas of proving the decidability of the monadic fragments of logics defined by a single finite Kripke frame were transferred to logics with equality. As a result, it was possible to prove the decidability of their monadic fragments with equality under different understandings of equality; similar results were obtained for polymodal and superintuitionist logics [2, 71].
- In addition, generalizations of the Kripke construction [31] have been obtained, which allows us to simulate binary predicate letters of a language using modality and unary predicate letters [72].

5. Publications submitted defense

Seven publications are submitted for the defense of the dissertation. All of them are devoted to non-classical logics and contain results related to the algorithmical complexity of such logics and their fragments.

1. Rybakov M., Shkatov D. Complexity of finite-variable fragments of propositional modal logics of symmetric frames. Logic Journal of the IGPL, 27(1):60–68, 2019.
   DOI: 10.1093/jigpal/jzy018
2. Rybakov M. N. Complexity of intuitionistic propositional logic and its fragments. Journal of Applied Non-Classical Logics, 18(2–3):267–292, 2008.
   DOI: 10.3166/jancl.18.267-292
3. Rybakov M., Shkatov D. Complexity and expressivity of propositional dynamic logics with finitely many variables. Logic Journal of the IGPL, 26(5):539–547, 2018.
   DOI: 10.1093/jigpal/jzy014

4. Rybakov M., Shkatov D. Complexity of finite-variable fragments of products with non-transitive modal logics. Journal of Logic and Computation, 32(5):853–870, 2022.
DOI: 10.1093/logcom/exab080
5. Rybakov M., Shkatov D. Complexity function and complexity of validity of modal and superintuitionistic propositional logics. Journal of Logic and Computation, 33(7):1566–1595, 2023.
DOI: 10.1093/logcom/exac085
6. Rybakov M. N. Undecidability of modal logics of a monadic predicate. Logical Investigations, 23(2):60–75, 2017. (In Russian)
DOI: 10.21146/2074-1472-2017-23-2-60-75
7. Rybakov M., Shkatov D. Algorithmic properties of modal and superintuitionistic logics of monadic predicates over finite Kripke frames. Journal of Logic and Computation, 35(2):exad078, 2025.
DOI: 10.1093/logcom/exad078
8. Rybakov M., Shkatov D. Algorithmic properties of first-order modal logics of finite Kripke frames in restricted languages. Journal of Logic and Computation, 30(7):1305–1329, 2020.
DOI: 10.1093/logcom/exaa041
9. Rybakov M., Shkatov D. Algorithmic properties of first-order superintuitionistic logics of finite Kripke frames in restricted languages. Journal of Logic and Computation, 31(2):494–522, 2021.
DOI: 10.1093/logcom/exaa091
10. Rybakov M. Predicate counterparts of modal logics of provability: High undecidability and Kripke incompleteness. Logic Journal of the IGPL, 32(3):465–492.
DOI: 10.1093/jigpal/jzad002

Conclusion

Summarizing the observations made regarding the algorithmic properties of non-classical logics, we obtain the following:

- in many monomodal and polymodal propositional logics, propositional variables can be simulated by veriable-free or one-variable formulas;
- in many superintuitionistic propositional logics, propositional variables can be simulated by two-variable formulas;
- the complexity of the satisfiability problem for a propositional logic and the complexity function for it are not directly related, even for fragments with a finite number of

propositional variables;

- modal and superintuitionistic predicate logics are usually undecidable in a language with a single unary letter and two individual variables;
- modal and superintuitionistic predicate logics of classes of finite Kripke frames, as a rule, are not recursively enumerable in a language with a single unary predicate letter and three individual variables;
- modal predicate logics defined by infinite ordinals and classes of Noetherian orders are usually Π^1_1-hard in a language with two unary predicate letters and two individual variables;
- modal prediacate logics defined only by non-elementary classes of Kripke frames can be recursively enumerable.

At the same time, the thesis describes general methods for simulating predicate letters of a language with formulas containing a single unary predicate letter in modal and superintuitionistic logics.

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